If AC, AD, AE, AF, BC, BD, BE, BF are homographic, the six points A, B,C,D,E,F, no three of them being on a straight line, lie on a curve of the second degree. Thus, by the preceding proposition, the rays drawn from any two of the six points A,B,C,D,E,F to the other four are homographic systems.

Chasles proves these propositions in the inverse order to that here given. See *Traité de Géométrie Supérieure*, Chapter XXV, Arts. 544, 416.

SOLUTIONS OF EXERCISES.

9

It is assumed that when a gate in a water-pipe is closing the pressure increases uniformly and the discharge decreases uniformly. Investigate an expression for the shortest safe time for closing the gate on the basis of these hypotheses: given the length of the pipe, the velocity of the stream, the working pressure, and the greatest admissible pressure to which the pipe may be exposed.

[*W. M. Thornton.*]

SOLUTION.

Put L for the length of the pipe,
V velocity of the stream,
Ω sectional area of the pipe,
g acceleration due to gravity,
D heaviness of water,
p working pressure in the pipe,
P highest admissible pressure in the pipe,
T required time in seconds.

The kinetic energy of the stream is

$$\frac{D}{2g} \cdot L\Omega V^2.$$

The velocity and pressure at t seconds after the closing of the valve begins are

$$V_t = V\left[1 - \frac{t}{T}\right], \qquad p_t = p + (P - p)\frac{t}{T};$$

and the work done by the stream in entering this region of high pressure is

$$\Omega\int_0^T P_t V_t\,dt = \Omega V\int_0^T \left(1 - \frac{t}{T}\right)\left[p + (P - p)\frac{t}{T}\right]dt$$

$$= \frac{1}{6}\,\Omega V(P + 2p)\,T.$$

If we neglect the work done against pipe friction we may equate the kinetic energy to the work done. We get thus for the required time

$$T = \frac{D}{g}\cdot\frac{3LV}{P + 2p};$$

and conversely for the water-ram produced by closing the gate in T seconds

$$P - p = \frac{D}{g}\cdot\frac{3LV}{T} - 3p.$$

[*Wm. M. Thornton.*]

19

SHOW geometrically (without using the calculus) that the asymptote of the hyperbolic spiral, $r = a : \theta$, is parallel to the initial line and distant a above it.

[*L. G. Barbour.*]

SOLUTION.

The Cartesian co-ordinates of a point on the spiral are

$$x = a\frac{\cos\theta}{\theta}, \qquad y = a\frac{\sin\theta}{\theta}.$$

When θ decreases without limit x increases without limit, but y converges to limit a. Hence $y = a$ is asymptotic to the spiral. [*L. G. Barbour.*]

26

DETERMINE the maximum right cone B inscribed in a given right cone A, the vertex of B being at the centre of the base of A. [*O. Root, Jr.*]

SOLUTION.

If the altitude of A be a and that of B be $a - x$, the area a of the base of B will be proportional to x^2, and its volume to $y = x^2(a - x)$; and for the maximum

$$\frac{dy}{dx} = 2ax - 3x^2 = 0;$$

$$\therefore\ x = \frac{2}{3}a, \text{ and } a - x = \frac{1}{3}a.$$

[*T. U. Taylor.*]

28

GIVEN the perpendicular, median, and bisector issuing from one and the same vertex of a plane triangle and terminating in the opposite side, to construct the triangle and determine a formula for its area. [*Marcus Baker.*]

SOLUTION.

1. *Construction of the triangle* This part of the problem is old.

2. *Calculation of the area.* Let p,q,r be the given perpendicular, bisector, and median, and P,Q,R their points of termination. Put l,m,n for QR, RP, PQ respectively and $2x$ for the base. Then the area is px and x is determined from the condition [*Euc.* VI, 3]

$$\frac{x+l}{x-l}=\frac{\sqrt{[p^2+(x+m)^2]}}{\sqrt{[p^2+(x-m)^2]}},$$

whence
$$x^2=\frac{lr^2-ml^2}{n};$$

$$\therefore\ \Delta=p\sqrt{\frac{lr^2-ml^2}{n}}.$$

Or, as will be found after obvious reductions,

$$\Delta=p\sqrt{\left[(2p^2-q^2)\sqrt{\frac{r^2-p^2}{q^2-p^2}}-(2p^2-r^2)\right]}.$$

[*W. M. Thornton.*]

29

GIVEN a pair of points A,B, if C,D are such that $OD\,.\,OC=OA^2=OB^2$ and $AOC=AOD$, prove that the pair A,B bears similar relation to the pair C,D. Show, also, the existence of a pair E,F which bears the same relation to each of the pairs A,B and C,D. [*Wm. Woolsey Johnson.*]

SOLUTION.

1. Assume O to be the middle point of AB. As an easy deduction from the conditions given, the triangles COA and AOD are similar; also the triangles COB and BOD.

$$\angle CAD=\angle CAO+\angle OCA=\angle COB=\angle BOD$$

and
$$\angle CBD=\angle CBO+\angle OCB=\angle COA=\angle AOD;$$

also
$$(OB=)\,AO:CB::OD:BD$$

and
$$(OA=)\,BO:CA::OD:AD;$$

$\therefore$ the triangles AOD and CBD are similar;

also the triangles BOD and CAD are similar.

If P be a point in CD such that $\angle BAD = \angle CAP$, since $\angle OBD = \angle ACD$, the triangles BAD and CAP are similar.

Dividing $OB\,(= AO) : OD :: AC : AD :: CB : BD$

by $$AP : AD :: CP : BD,$$

we have $$AC : AP :: CB : CP;$$

and, since $\angle OAD = \angle CAP = \angle BCD$, the triangles CAP and BCP are similar, and

$$AP : CP :: CP : BP, \text{ or } AP \times BP = CP^2;$$

similarly it may be shown that $AP \times BP = DP^2$.

2. Let $C'A$ be a line parallel to BD through A, and Q a point such that $\angle BQC = \angle C'AC$ and $BQ : CQ :: BD : CA$; then the triangles BDQ and CAQ will be similar and also the triangles CQB and AQD.

Let FQ be a line bisecting $\angle AQB$ and put $FQ^2 = AQ \times BQ$; continue FQ to the point E such that $QE = FQ$; then will E and F satisfy the conditions required. [*Ormond Stone.*]

31

In the triangle ABC draw AD to the point D in BC; then will

$$AB^2 . DC + AC^2 . DB = BC . AD^2 + BD . DC . BC.$$

[*J. R. Spiegel.*]

SOLUTION.

From C and B draw CE and BF meeting AD at right angles at E and F. Since ED and DF are the projections of BD and DC respectively on AD, we have

$$AB^2 . DC = AD^2 . DC + BD^2 . DC + 2AD . ED . DC,$$
$$AC^2 . BD = AD^2 . BD + CD^2 . BD - 2AD . DF . BD.$$

Adding, we obtain the required relation, since $DC + BD = BC$, and, by similar triangles, $ED . DC = DF . BD$. [*A. V. Lane.*]